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FILTRATION OF FLUID IN CURVILINEAR LAYERS OF VARIABLE THICKNESS*

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An effective solution (in many cases in quadratures) of the boundary value problems of fluid filtration theory in thin curvilinear layers of variable thickness is given. The main idea of the approach adopted is to replace the initial system of equations of two-dimensional filtration of fluid by another, simpler system of equations, equivalent to the first system within the limits of its accuracy. An example is given.

1. General approach. Let us consider the filtration of an incompressible heavy fluid in a thin, curvilinear layer of porous material contained between two impermeable surfaces in three-dimensional space. We denote by α and β the Gaussian orthogonal coordinates of the middle surface of the curvilinear layer, chosen in such a manner that the coordinate lines are the lines of principal curvatures of this surface. Further we denote by γ the normal to the middle surface (so that $\alpha\beta\gamma$ form a right coordinate system), and by h the layer thickness, the latter being a known function of α and β. We shall assume that

$$\frac{1}{A}\frac{\partial h}{\partial \alpha} \ll 1, \quad \frac{1}{B}\frac{\partial h}{\partial \beta} \ll 1 \tag{1.1}$$

Here A and B are the coefficients of the first quadratic form of the middle surface in the coordinates α and β.

Under these assumptions the equations of two-dimensional theory of filtration have the following form:

$$\frac{\partial (hv_\alpha)}{\partial \alpha} + \frac{\partial (hv_\beta)}{\partial \beta} = 0 \tag{1.2}$$

$$v_\alpha = -\frac{k}{\mu A}\left(\frac{\partial p}{\partial \alpha} + \rho g \frac{\partial z}{\partial \alpha}\right), \quad v_\beta = -\frac{k}{\mu B}\left(\frac{\partial p}{\partial \beta} + \rho g \frac{\partial z}{\partial \beta}\right) \tag{1.3}$$

where p is the pressure of the fluid, v_α and v_β are the filtration rate components along the α and β axes, μ is the dynamic viscosity of the fluid, k is the permeability of the porous material, ρ is the fluid density and g is acceleration due to gravity acting in the negative direction of the z-axis. The equation of the middle surface of the layer in the Cartesian xyz-coordinates has the form

$$x = x(\alpha, \beta), \; y = y(\alpha, \beta), \; z = z(\alpha, \beta) \tag{1.4}$$

Substituting (1.3) into (1.2), we obtain a single equation for the function $p(\alpha, \beta)$

$$\frac{\partial}{\partial \alpha}\left(\frac{h}{A}\frac{\partial p}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{h}{B}\frac{\partial p}{\partial \beta}\right) = -\rho g\left[\frac{\partial}{\partial \alpha}\left(\frac{h}{A}\frac{\partial z}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{h}{B}\frac{\partial z}{\partial \beta}\right)\right] \tag{1.5}$$

Assuming that the derivatives (1.1) are small, we can write the expression for the layer thickness in the following form:

$$h = h_0 + \varepsilon h_1(\alpha, \beta) \quad (\varepsilon \ll 1) \tag{1.6}$$

where ε is a small number, h_0 is a constant and h_1 is a known function of α and β.

The equations (1.2) and (1.3), and hence (1.5), hold only when the derivatives (1.1) are small. Near the boundaries of the layer and in the regions where its thickness varies simply, an edge effect appears which cannot be examined in the two-dimensional approximation and demands therefore, in one way or another, the introduction of the three-dimensional equations of the filtration theory. At the distances from the layer edges equal to several times the layer

*Prikl.Matem.Mekhan.,45,No.6,1142-1146,1981 thickness, the exact three-dimensional solution tends asymptotically to the approximate two-dimensional solution. The condition of smallness of the derivative mentioned above is obviously equivalent to the equation (1.6), i.e. to the existence of a small number ε.

It is easily seen that the exact solution of the system (1.2), (1.3) differs from the exact solution of the corresponding three-dimensional problem of the theory of filtration, at a distance from the layer edge, by a quantity of the order of ε^2. An error of this magnitude appears in the formulation of the D'Arcy equations (1.3) for a layer of variable thickness. This can be verified directly by analyzing the exact solutions of the three-dimensional theory of filtration for the wedge-like and cone-like layers. Consequently the solution of the system of equations (1.2), (1.3) describes the state of the physical system (filtration in a layer of variable thickness), approximately, with an error of the order of ε^2. We note that the quantity v_γ which does not appear in (1.2) and (1.3), is of the order of ε.

Let us formulate the following problem. Is it not possible to find a new, simpler system of equations equivalent to the initial system (1.2), (1.3), within the limits of its accuracy in the sense that the solution of the new system will differ from the solution sought by a quantity of the order of ε^2. Obviously, in this case we can call the solution of the new system an "exact" solution of the initial system insofar as its accuracy matches the accuracy of the initial system. We note that this concept can also be used in any approximate numerical solutions of the boundary value problems of mathematical physics the too high accuracy of which is often unjustified by the approximate character of the initial equations.

In the present case the equivalent system of equations is easily constructed using the method of small parameter. Suppose that we wish to obtain a solution of some particular boundary value problem for equation (1.5), assuming that such solution exists and is unique. We seek the solution of the problem in the form

$$p = p_0(\alpha, \beta) + \varepsilon p_1(\alpha, \beta) \tag{1.7}$$

Here the unknown functions p_0 and p_1 are independent of ε. We substitute the functions p and h given by (1.6) and (1.7) into (1.5), and equate the coefficients accompanying like powers of the small parameter ε, neglecting terms of the order of ε^2 and higher, the latter exceeding the limits of accuracy of the initial equations. In the more general cases this simple procedure fails, since the small parameter accompanies the leading derivative. As a result we have

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial p_0}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial p_0}{\partial\beta}\right) = -\rho g\left[\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial z}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial z}{\partial\beta}\right)\right] \tag{1.8}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial p_1}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial p_1}{\partial\beta}\right) = -\frac{1}{h_0A}\frac{\partial h_1}{\partial\alpha}\left(\frac{\partial p_0}{\partial\alpha} + \rho g\frac{\partial z}{\partial\alpha}\right) - \frac{1}{h_0B}\frac{\partial h}{\partial\beta}\left(\frac{\partial p_0}{\partial\beta} + \rho g\frac{\partial z}{\partial\beta}\right) \tag{1.9}$$

Thus a system of equations equivalent to the initial equation (1.5) consists of two, similar type equations (1.8) and (1.9), differing from each other only in their right-hand parts. The first equation becomes identical to (1.5) when $h = h_0 = \text{const}$, and the right-hand side of the second equation is determined by the solution of the first equation. The boundary conditions for these equations can be formulated with help of the boundary value problem. The fact that the resulting formulation is not unique, is clearly immaterial in determining the physically meaningful function $p(\alpha, \beta)$.

The solutions of the boundary value problems for the system of equations (1.8), (1.9) can be obtained in quadratures, if the function generating the corresponding boundary value problem for the equation

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial u}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial u}{\partial\beta}\right) = 0$$

is known. The generating function for this equation can be found analytically for many cases of practical interest (plane, cylindrical and conical surfaces, sphere, "hollow" surfaces, etc.). In all these cases the principle of superposition can be used to write the solution of the initial boundary value problem for a curvilinear layer of variable thickness in the form of quadruple (double in simpler cases) integrals. The analytic solutions are of great practical interest, in particular in connection with the possibilities emerging for the subsequent effective solutions of the optimization problems.

2. Plane layers of variable thickness. In the case of plane layers of variable thickness the equation (1.5) has the following form in the Cartesian xy-coordinates:

$$\frac{\partial}{\partial x}\left(h\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial p}{\partial y}\right)=-\rho g\left(\cos\varphi\frac{\partial h}{\partial x}+\cos\psi\frac{\partial h}{\partial y}\right) \tag{2.1}$$

Here φ and ψ denote the angles formed by the z-axis with the x and y axes respectively (this time the x and y axes lie in the plane of the layer, and the z-axis is defined, as before, by the direction of the gravity force). A system of equations equivalent to the above equation has the form

$$\frac{\partial^2 p_0}{\partial x^2}+\frac{\partial^2 p_0}{\partial y^2}=0 \tag{2.2}$$

$$\frac{\partial^2 p_1}{\partial x^2}+\frac{\partial^2 p_1}{\partial y^2}=-\frac{1}{h_0}\frac{\partial h_1}{\partial x}\left(\frac{\partial p_0}{\partial x}+\rho g\cos\varphi\right)-\frac{1}{h_0}\frac{\partial h_1}{\partial y}\left(\frac{\partial p_0}{\partial y}+\rho g\cos\psi\right),\quad (p=p_0+\varepsilon p_1) \tag{2.3}$$

Thus the case in question has been reduced to two problems of the theory of harmonic potential, which can be solved one after the other. It follows therefore that many boundary value problems for (2.1) can be solved efficiently in quadratures using the proposed method. Approximately the same level of difficulty is encountered in the corresponding boundary value problems for the cylindrical, conical and generally developable surfaces, in connection with the fact that in this case the coefficients A and B will be constant.

We see that the proposed method is essentially different from the standard method of small parameter, the difference being as follows. The method of small parameter is an approximate method of solving "exact" equations, and the solution is obtained in the form of an infinite series in powers of the small parameter. The proposed method is an "exact" method of solving the same initial equations which are, however, already regarded as approximate. Estimation of the error present in these equations leads to conclusion that the "exact" solution has the form (1.7), i.e. it formally coincides with the first two terms of the expansion in terms of the small parameter.

The standard method of small parameter is well known in the theory of filtration (see /1/). The flow of fluid in curvilinear layers was studied by many authors, with Golubeva in /2/ giving the greatest attention to these problems.

3. Example. Let the middle surface of the layer be described by the following equations:

$$z=-\frac{ba^2}{\beta^2+a^2}\quad (b>0),\qquad x=\int_0^\beta\frac{[(\beta^2+a^2)^4-4\beta^2b^2a^4]^{1/2}}{(\beta^2+a^2)^2}\,d\beta \tag{3.1}$$

Here a and b are positive parameters of the dimension of length. Equations (3.1) describe a cylindrical surface with the generatrices parallel to the y-axis. The transverse cross section of this surface represents a symmetric, lune-like curve in the Oxz plane, tending asymptotically to the x-axis as $x\to\pm\infty$ (see Fig.1). The variable β clearly represents the arc length of this curve counted from the minimum point $x=0, z=-b$. The other orthogonal coordinate of the middle surface (α) can be regarded, without loss of generality, as equal to y. The coefficients A and B will be equal to unity.

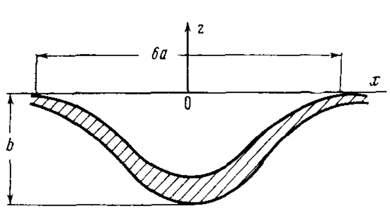

Fig.1

The equations (3.1) approximate sufficiently accurately the form of a geological fold containing oil and gas deposits, provided that we choose the greatest depth of the fold as b, and approximately one sixth of the fold width as a (and replacing z by $-z$). Let the layer thickness be defined by

$$h=h_0\left(1+\frac{\varepsilon a^2}{\beta^2+a^2}\right)$$

Assume that a heavy fluid saturates the whole layer $-\infty<\alpha<+\infty, -\infty<\beta<+\infty$ and a sink of the fluid of strength Q (the well) is present at the point $x=0, y=0, z=-b$, (i.e. at the coordinate origin of the $\alpha\beta$-plane). We require to determine the pressure and filtration rate fields in the layer. In the present case the pressure is given either by a single equation

$$\frac{\partial}{\partial\alpha}\left(h\frac{\partial p}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(h\frac{\partial p}{\partial\beta}\right)=-\rho g\frac{\partial}{\partial\beta}\left(h\frac{\partial z}{\partial\beta}\right) \tag{3.2}$$

or by the equivalent system of two equations

$$\Delta p_0=-\rho g\frac{\partial^2 z}{\partial\beta^2}\quad\left(\Delta=\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right) \tag{3.3}$$

$$\Delta p_1=-\frac{1}{h_0}\frac{\partial h_1}{\partial\beta}\left(\frac{\partial p_0}{\partial\beta}+\rho g\frac{\partial z}{\partial\beta}\right),\quad\left(h_1=\frac{h_0a^2}{\beta^2+a^2},\quad p=p_0+\varepsilon p_1\right)$$

It is doubtful whether a solution of (3.2) can be obtained in analytic form, in contrast to the solution of system (3.3). The Green's function for the Laplace equation has the form $\ln[(\alpha-\alpha_0)^2+(\beta-\beta_0)]^{1/2}$ where α_0 and β_0 are the coordinates of the source (or sink). Hence we can write the solution of system (3.3) in the following form

$$p_0=-\frac{\mu Q}{2\pi k}\ln(\alpha^2+\beta^2)^{1/2}+\frac{\rho gba^2}{\beta^2+a^2}+p_\infty \tag{3.4}$$

$$p_1=-\frac{\mu Qa^2}{\pi k}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\beta_0^2\ln[(\alpha-\alpha_0)^2+(\beta-\beta_0)^2]^{1/2}}{(\alpha_0^2+\beta_0^2)(a^2+\beta_0^2)^2}d\alpha_0\,d\beta_0$$

$$(p=p_0+\varepsilon p_1)$$

where p_∞ is the pressure in the fluid at $Q=0$ and $\beta\to\pm\infty$.

The above solution can be regarded as an exact solution of equation (3.2) in the sense explained above.

REFERENCES

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